Existence of Simultaneous Route and Departure Choice Dynamic User Equilibrium

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Abstract

This paper is concerned with the existence of the *simultaneous route-and-departure choice dynamic user equilibrium* (SRDC-DUE) in continuous time. The SRDC-DUE problem was formulated as an infinite-dimensional variational inequality in Friesz et al. (1993). In deriving our existence result, we employ the *generalized Vickrey model* (GVM) introduced in Han et al. (2012a, 2012b) to formulate the underlying network loading problem. As we explain, the GVM corresponds to a path delay operator that is provably strongly continuous on the Hilbert space of interest. Finally, we provide the desired SRDC-DUE existence result for general constraints relating path flows to a table of fixed trip volumes without invocation of a priori bounds on the path flows.

1 Introduction

In this paper we shall consider *dynamic traffic assignment* (DTA) to be the positive (descriptive) modeling of time-varying flows of automobiles on road networks consistent with established traffic flow theory and travel demand theory. *Dynamic User Equilibrium* (DUE) is one type of DTA wherein effective unit travel delay for the same purpose is identical for all utilized path and departure time pairs. The relevant notion of travel delay is effective unit travel delay, which is the sum of arrival penalties and actual travel time. For our purposes in this paper, DUE is modeled for the within-day time scale based on fixed travel demands established on a day-to-day time scale.

In the last two decades there have been many efforts to develop a theoretically sound formulation of dynamic network user equilibrium that is also a canonical form acceptable to scholars and practitioners alike. DUE models tend to be comprised of four essential sub-models:

1. a model of path delay;

2. flow dynamics;

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3. flow propagation constraints;

4. a path/departure-time choice model.

Furthermore, analytical DUE models tend to be of two varieties: (1) *route choice* (RC) user equilibrium (Friesz et al. 1989, Merchant and Nemhauser 1978a, 1978b, Mounce 2006, Smith and Wisten 1995, Zhu and Marcotte 2000) ; and (2) *simultaneous route-and-departure choice* (SRDC) dynamic user equilibrium (Friesz et al. 1993, 2010, 2012, Ran et al. 1996, Wie et al. 2002). For both types of DUE models, the existence of a dynamic user equilibrium in continuous time remains a fundamental issue. A proof of DUE existence is a necessary foundation for qualitative analysis and computational studies. In this paper, we provide a DUE existence result for the SRDC DUE problem when it is formulated as an infinite-dimensional variational inequality of the type presented in Friesz et al. (1993). In this paper, in order to establish a DUE existence result, we study the network loading problem based on the generalized Vickrey model (GVM) proposed in Han et al. (2012a, 2012b). All of our results presented in this paper are more general than any obtained previously for DUE when some version of the point queue model is employed.

1.1 Formulation of the SRDC user equilibrium

There are two essential components within the RC or SRDC notions of DUE: (i) the mathematical expression of Nash-like equilibrium conditions, and (ii) a network performance model, which is, in effect, an embedded network loading problem. The embedded network loading problem captures the relationships among arc entry flow, arc exit flow, arc delay and path delay for any path departure rate trajectory. Note that, by studying the embedded network loading problem based on the GVM, we are not suggesting or employing a sequential solution paradigm for DUE. If the reader believes a sequential perspective is implicit in our work, he/she has not fully understood the mathematical presentation provided in this and subsequent sections.

There are multiple means of expressing the Nash-like notion of a dynamic equilibrium, including the following:

1. a variational inequality (Friesz et al. 1993; Smith and Wisten 1994, 1995)

2. an equilibrium point of an evolution equation in an appropriate function space (Mounce 2006; Smith and Wisten 1995)

3. a nonlinear complementarity problem (Wie et al. 2002; Han et al. 2011)

4. a differential variational inequality (Friesz et al. 2001, 2010, 2012; Friesz and Mookherjee 2006); and

5. a differential complementarity system (Pang et al. 2011).

The variational inequality representation is presently the primary mathematical form employed for both RC and SRDC DUE. The most obvious approach to establishing existence for any of the mathematical representations mentioned above is to convert the problem to an equivalent fixed point problem and then apply Brouwer's fixed point existence theorem. Alternatively, one may use an existence theorem for the particular mathematical representation selected; it should be noted that most such theorems are derived by using Brouwer's famous theorem. So, in effect, all proofs of DUE existence employ Brouwer's fixed point theorem, either implicitly

or explicitly. One statement of Brouwer's theorem appears as Theorem 2 of Browder (1968). Approaches based on Brouwer's theorem require the set of feasible path flows (departure rates) under consideration to be compact and convex in a Banach space, and typically involve an a priori bound on each path flow.

We also wish to point out that this paper employs much more general constraints relating path flows to a table of fixed trip volumes than has been previously considered when studying SRDC DUE. Moreover, in our study of existence, we do not invoke a priori bounds on the path flows to assure boundedness needed for application of Brouwer's theorem. That is, a goal of this paper is to investigate the existence of DUE without making the assumption of *a priori* bounds for departure rates. Note should be taken of the following fact: the boundedness assumption is less of an issue for the RC DUE by virtue of problem formulation; that is, for RC DUE, the travel demand constraints are of the following form:

$$\sum_{p\in\mathcal{P}_{ij}} h_p(t) \;= R_{ij}(t), \qquad \forall\, t, \quad \forall\,(i,\,j)\in\mathcal{W} \tag{1.1}$$

where $\mathcal{W}$ is the set of origin-destination pairs, $\mathcal{P}_{ij}$ is the set of paths connecting $(i,j)\in\mathcal{W}$ and $h_p(t)$ is the departure rate along path p. Furthermore, $R_{ij}(t)$ represents the rate (not volume) at which travelers leave origin i with the intent of reaching destination j at time t; each such trip rate is assumed to be bounded from above. Since (1.1) is imposed pointwise and every path flow h_p is nonnegative, we are assured that each $h=(h_p:(i,j)\in\mathcal{W},p\in\mathcal{P}_{ij})$ are automatically uniformly bounded. On the other hand, the SRDC user equilibrium imposes the following constraints on path flows:

$$\sum_{p\in\mathcal{P}_{ij}}\int_{t_0}^{t_f} h_p(t)\,dt \;=\; Q_{ij}, \qquad \forall\,(i,\,j)\in\mathcal{W} \tag{1.2}$$

where $Q_{ij}\in\Re^1_+$ is the volume (not rate) of travelers departing node i with the intent of reaching node j. The integrals in (1.2) are interpreted as Legesgue; hence, (1.2) alone is not enough to assure bounded path flows. This observation has been the major hurdle to proving existence without the a priori invocation of bounds on path flows. In this paper, we will overcome this difficulty through careful analysis of the GVM and by investigating the effect of user behavior in shaping network flows, in a mathematically intuitive yet rigorous way.

1.2 Importance of the path delay operator

Clearly another key component of continuous-time DUE is the path delay operator, typically obtained from *dynamic network loading* (DNL), which is a subproblem of a complete DUE model[1]. Any DNL must be consistent with the established path flows and link delay model, and DNL is usually performed under the *first-in-first-out* (FIFO) rule. The properties of the delay operator are critical to proving existence of a solution to the infinite-dimensional variational inequality used to express DUE. In Zhu and Marcotte (2000), using the *link delay model* introduced by Friesz et al. (1993), the authors showed weak continuity of the path delay

[1]Note that, by refering to the network loading procedure, we are neither employing nor suggesting a sequential approach to the study and computation of DUE. Rather a subset of the equations and inequalites comprising a complete DUE model may be grouped in a way that identifies a traffic assignment subproblem and a network loading subproblem. Such a grouping and choice of names is merely a matter of convenient language that avoids repetitive reference to the same mathematical expressions. Use of such language does not alter the need to solve both the assignment and loading problems consistently and, thus, simultaneously. A careful reading of the mathematical presentation made in subsequent sections makes this quite clear.

operator under the assumption that the path flows are *a priori* bounded. Their continuity result is superceded by a more general result proven in this paper: the path delay operator of interest is strongly continuous without the assumption of boundedness. Strong continuity without boundedness is central to our proof of existence in the present paper.

In this paper, as a foundation for DNL, we will consider Vickrey's model of congestion first introduced in Vickrey (1969) and later studied by Han et al. (2012a, 2012b). Vickrey's model for one link is primarily described by an *ordinary differential equation* (ODE) with discontinuous right hand side. Such irregularity has made it difficult to analyze Vickery's model in continuous time. Fortunately, in this paper, we will be able to take advantage of the reformulation proposed in Han et al. (2012a, 2012b), then prove the strong continuity of the path delay operator without boundedness of the path flows. This will provide a quite general existence proof for SRDC-DUE based on the GVM.

1.3 Organization

The balance of this paper is organized as follows. Section 2 provides essential mathematical background on the concepts that will be used in the paper. Section 3 briefly reviews the formal definition of dynamic user equilibrium and its formulation as a variational inequality. Section 4 recaps the generalized Vickrey model (GVM) originally put forward by Han et al. (2012a, 2012b). Section 5 formally discusses the properties of the effective delay operator. The main result of this paper, the existence of an SRDC-DUE when the GVM informs network loading is established in Theorem 5.7 of Section 5.3.

2 Mathematical preliminaries

A *topological vector space* is one of the basic structures investigated in functional analysis. Such a space blends a topological structure with the algebraic concept of a vector space. The following is a precise definition:

Definition 2.1. ***(Topological vector space)*** *A topological vector space X is a vector space over a topological field $\mathbb{F}$ (usually the real or complex numbers with their standard topologies) which is endowed with a topology such that vector addition $X \times X \to X$ and scalar multiplication $\mathbb{F} \times X \to X$ are continuous functions.*

As a consequence of Definition 2.1, all normed vector spaces, and therefore all Banach spaces and Hilbert spaces, are examples of topological vector spaces. Also important is the notion of a seminorm:

Definition 2.2. ***(Seminorm)*** *A seminorm on a vector space X is a real-valued function p on X such that*

(a) $p(x+y) \;\leq\; p(x)+p(y)$

(b) $p(\alpha\, x) \;=\; |\alpha|\, p(x)$

for all x and y in X and all scalars α.

Definition 2.3. ***(Locally convex space)*** *A locally convex space is defined to be a vector space X along with a family of seminorms $\{p_i\}_{i\in\mathcal{I}}$ on X.*

As part of our review we make note of the following essential knowledge:

Fact 1. The space of square-integrable real-valued functions on a compact interval $[a, b]$, denoted by $\mathcal{L}^2([a, b])$, is a locally convex topological vector space.

Fact 2. The m-fold product of the spaces of square-integrable functions $\mathcal{L}^2([a, b])^m$ is a locally convex topological vector space.

Definition 2.4. ***(Dual space)*** *The dual space X^* of a vector space X is the space of all continuous linear functions on X*

Another key property we consider without proof is:

Fact 3. The dual space of $L^p(\mu)$ for $1 < p < \infty$ has a natural isomorphism with $L^q(\mu)$ where q is such that $1/p + 1/q = 1$. In particular, the dual space of $\mathcal{L}^2([a, b])$ is again $\mathcal{L}^2([a, b])$.

The key foundation for analysis of existence is the following theorem given in Browder (1968):

Theorem 2.5. *Let K be a compact convex subset of the locally convex topological vector space E, T a continuous (single-valued) mapping of K into E^*. Then there exits u_0 in K such that*

$$\Big\langle T(u_0),\, u_0 - u \Big\rangle \;\geq\; 0$$

for all $u \in K$.

Let us now give the formal definition of a variational inequality in a topological setting:

Definition 2.6. ***(Infinite-Dimensional Variational inequality)*** *Let V be a topological vector space and $F : U \times \Re^1_+ \to V$, where $U \subset V$. The infinite-dimensional variational inequality is posed as the following problem*

$$\left.\begin{array}{c} \text{find } u^* \in U \\ \langle F(u),\, u - u^* \rangle \;\geq\; 0 \;\; \forall\, u \in U \end{array}\right\} \; VI(F, U) \tag{2.3}$$

Definition 2.7. ***(Compactness of subspaces)*** *A subset K of a topological space X is called compact if for every arbitrary collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X such that*

$$K \;\subset\; \bigcup_{\alpha \in A} U_\alpha$$

there is a finite subset I of A such that

$$K \;\subset\; \bigcup_{i \in I} U_i$$

Definition 2.8. ***(Sequential compactness)*** *A topological space is sequentially compact if every sequence has a convergent subsequence.*

An outgrowth of the concepts and results given above, the following fact is stated without proof:

Fact 4. In metric space (hence topological vector space), the notions of compactness and sequential compactness are equivalent.

The final bit of specialized knowledge about topological vector spaces that we shall need is the following:

Definition 2.9. ***(Weak convergence in Hilbert space)*** *A sequence of points $\{x_n\}$ in a Hilbert space $\mathcal{H}$ is said to be convergent weakly to a point $x \in \mathcal{H}$, denoted as $x_n \rightharpoonup y$ if*

$$\langle x_n,\, y \rangle \;\;\to\;\; \langle x,\, y \rangle$$

for all $y \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle$ is the inner product on the Hilbert space.

3 Continuous-time dynamic user equilibrium

In this section, we will assume the time interval of interest is

$$[t_0,\, t_f] \subset \Re^1_+$$

The most crucial component of the DUE model is the path delay operator, which provides the time to traverse any path p per unit of flow departing from the origin of that path. The delay operator is denoted by

$$D_p(t,\, h) \qquad \forall\; p \in \mathcal{P}$$

where $\mathcal{P}$ is the set of all paths employed by network users, t denotes the departure time, and h is a vector of departure rates. Throughout the rest of the paper, we stipulate that

$$h \in \Big(\mathcal{L}^2_+([t_0,\, t_f]\Big)^{|\mathcal{P}|}$$

where $\Big(\mathcal{L}^2_+[t_0,\, t_f]\Big)^{|\mathcal{P}|}$ denotes the non-negative cone of the $|\mathcal{P}|$-fold product of the Hilbert space $\mathcal{L}^2[t_0,\, t_f]$ of square-integrable functions on the compact interval $[t_0,\, t_f]$. The inner product of the Hilbert space $\Big(\mathcal{L}^2[t_0,\, t_f]\Big)^{|\mathcal{P}|}$ is defined as

$$\big\langle u,\, v\big\rangle \;\doteq\; \int_{t_0}^{t_f} \big(u(s)\big)^T v(s)\, ds \tag{3.4}$$

where the superscript T denotes transpose of vectors. Moreover, the norm

$$\big\|u\big\|_{\mathcal{L}^2} \;\doteq\; \big\langle u,\, u\big\rangle^{1/2} \tag{3.5}$$

is induced by the inner product (3.4).

Next, we need to consider a more general notion of travel cost that will motivate on-time arrivals. To this end, for each $p \in \mathcal{P}$, we introduce the effective unit path delay operator $\Psi_p : [t_0,\, t_f] \times \Big(\mathcal{L}^2_+([t_0,\, t_f])\Big)^{|\mathcal{P}|} \to \Re^1_{++}$ and define it as follows:

$$\Psi_p(t,\, h) \;\equiv\; D_p(t,\, h) + \mathcal{F}\Big(t + D_p(t,\, h) - T_A\Big) \tag{3.6}$$

where $\mathcal{F}(\cdot)$ is the penalty for early or later arrival relativerelative to the target arrival time T_A. Note that, for convenience, T_A is assumed to be independent of destination. However, that assumption is easy to relax, and the consequent generalization of our model is a trivial extension. We interpret $\Psi_p(t,\, h)$ as the perceived travel cost of driver starting at time t on path p under travel conditions h. Presently, our only assumption on such costs is that for each $h \in \Big(\mathcal{L}^2_+([t_0,\, t_f])\Big)^{|\mathcal{P}|}$, the vector function $\Psi(\cdot,\, h) : [t_0,\, t_f] \to \Re^{|\mathcal{P}|}_{++}$ is measurable and stictly positive. The assumption of measurability was used for a measure theory-based argument in Friesz et al. (1993). Later in this paper, we shall discuss other properties of this operator, such as continuity on a Hilbert space. The continuity of effective delay is crucial for applying the general theorems in Browder (1968), especially Theorem 2.5 stated above.

To support the development of a dynamic network user equilibrium model, we introduce some additional constraints. Foremost among these are the flow conservation constraints

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt \;=\; Q_{ij} \qquad \forall\; (i,\, j) \in \mathcal{W} \tag{3.7}$$

where $\mathcal{P}_{ij}$ is the set of all paths that connect origin-destination (O-D) pair $(i,\, j) \in \mathcal{W}$, while $\mathcal{W}$ is the set of all O-D pairs. In addition, Q_{ij} is the fixed travel demand for O-D pair $(i,\, j)$. Using the notation and concepts we have thus far introduced, the set of feasible solutions for DUE when the effective delay operator $\Psi(\cdot,\, \cdot)$ is given is

$$\Lambda \;=\; \left\{ h \in \Big(\mathcal{L}_+^2([t_0,\, t_f])\Big)^{|\mathcal{P}|} : \quad \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt \;=\; Q_{ij} \quad \forall\; (i,\, j) \in \mathcal{W} \right\} \tag{3.8}$$

Using a presentation very similar to the above, the notion of a dynamic user equilibrium in continuous time was first introduced by Friesz et al. (1993), who employ a definition tantamount to the following:

Definition 3.1. ***(Dynamic user equilibrium)****. A vector of departure rates (path flows) $h^* \in \Lambda$ is a dynamic user equilibrium if*

$$h_p^*(t) > 0, p \in \mathcal{P}_{ij} \Longrightarrow \Psi_p\left[t, h^*(t)\right] = v_{ij} \in \Re^1_{++} \qquad \forall (i,\, j) \in \mathcal{W} \tag{3.9}$$

We denote the dynamic user equilibrium defined this way by $DUE\left(\Psi, \Lambda, [t_0, t_f]\right)$.

In the analysis to follow, we focus on the following infinite-dimensional variational inequality formulation of the DUE problem reported in Theorem 2 of Friesz et al. (1993).

$$\left.\begin{array}{c} \text{find } h^* \in \Lambda \text{ such that} \\ \displaystyle\sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h_p^*)dt \geq 0 \\ \forall h \in \Lambda \end{array}\right\} VI\Big(\Psi, \Lambda, [t_0, t_f]\Big) \tag{3.10}$$

The variational inequality formulation $VI\big(\Psi,\, \Lambda, [t_0,\, t_f]\big)$ expressed above subsumes almost all DUE models regardless of the arc dynamics or network loading models employed provided $\Psi_p(t, h^*)$

4 The dynamic network loading

A key ingredient of the variational inequality formulation of the DUE (3.10) is the effective delay operator $\Psi(t,\, \cdot)$, which maps a vector of admissible departure rates to the vector of strictly travel costs associated with each route-and-departure-time choice. The problem of predicting time-varying network flows consistent with known travel demands and departure rates (path flows) is usually referred to as the *dynamic network loading* (DNL) problem. Since effective path delays are constructed from arc delays that depend on arc activity and performance, DNL is intertwined with the determination of effective delay operators.

In this section we present a continuous-time DNL model. This model is based on a reformulation of Vickrey's model (Vickrey 1969) that we refer to as the *generalized Vickrey model* (GVM); it was apparently first proposed in Han et al. (2012a, 2012b). The generalized Vickrey model determines arc exit flow and the arc traversal time from arc entry flow. This formulation not only leads to a simple and explicit computational scheme, but also makes it easier to conduct rigorous analyses of the arc delay operator and, hence, of the effective path delay operator $\Psi(t,\, \cdot)$.

4.1 The generalized Vickrey model

First introduced in Vickrey (1968), the Vickrey's model is based on two key assumptions: (i) vehicles have negligible sizes, and, therefore, any queue is of negligible size; and (ii) link travel time consists of a fixed travel time plus a congestion-related arc-traversal delay. Let us introduce the following notation:

$$
\begin{aligned}
u(t): &\quad \text{the flow profile into the link} \\
M: &\quad \text{the flow capacity of the link} \\
q(t): &\quad \text{the queue size} \\
w(t): &\quad \text{the link exit flow} \\
T: &\quad \text{the fixed free flow travel time} \\
D(t): &\quad \text{the link traverse time of drivers entering the link at } t
\end{aligned}
$$

Then the model is described by the following set of equations.

$$w(t) = \begin{cases} \min\{u(t-T),\, M\} & q(t) = 0 \\ M & q(t) \neq 0 \end{cases} \tag{4.11}$$

$$\frac{dq(t)}{dt} = u(t-T) - w(t) \tag{4.12}$$

$$D(t) = T + \frac{q(t+T)}{M} \tag{4.13}$$

Notice that (4.11) and (4.12) amount to an ordinary differential equation (ODE) with a right hand side that is discontinuous in the state variable $q(\cdot)$. Such ODEs have been the main hurdle to further analysis and computation of this model in continuous time. In Han et al. (2012a, 2012b), a reformulation of Vickrey's model as a Hamilton-Jacobi equation was proposed and solved with a version of the Lax-Hopf formula (see Evans 2010 for more detailed on general Hamilton-Jacobi equation and Lax-Hopf formula). As a result, the solution to (4.11)-(4.13) was presented in closed form. Due to the space limitation, we will omit details of the derivation of explicit solution to Vickrey's model and refer the reader to Han et al. (2012a, 2012b) for details.

Let us next introduce the cumulative entering vehicle count $U(\cdot)$ and the exiting vehicle count $\mathcal{W}(\cdot)$ at the entrance and exit of the link of interest, respectively. Furthermore, $U(\cdot)$ is assumed to be non-decreasing and left continuous. Notice that these latter assumption imply that the link entry flows can be unbounded and possibly contain the dirac delta function. In contrast, the Vickrey's original model requires that the entry flow to be at least Lebesgue integrable. As such, the GVM is more general than Vickrey's model.

Using the notation introduced previously, an equivalent statement of (4.11) and (4.12) is the following:

$$\mathcal{W}(t) = \min_{\tau \leq t-T} \{U(\tau) - M\,\tau\} + M(t - t_0) \tag{4.14}$$

$$q(t) = U(t) - M\,t - \min_{\tau \leq t-T} \{U(\tau) - M\,\tau\} \tag{4.15}$$

$$D(t) = T + \frac{q(t+T)}{M} \tag{4.16}$$

Note that all the unknowns of system (4.14)-(4.16) may be explicitly stated in terms of the cumulative vehicle count $U(\cdot)$. The system (4.14)-(4.16) will serve as the mathematical formulation of link dynamics in the DNL, as we shall explain shortly. The system (4.14)-(4.16) may also be used for deriving mathematical properties of the effective path delay operator, as is demonstrated in Section 5.1.

4.2 The network model

It is straightforward to extend the generalized Vickrey model to a network, which is represented as a directed graph $G(N,\,A)$, where N and A are the set of nodes and arcs, respectively. In order to proceed, we introduce some additional notations. In particular, for each node $v\in N$, let $\mathcal{I}^v$ be the set of incoming links, $\mathcal{O}^v$ the set of outgoing links. For each arc $a\in A$, let $u_a(t)$, $w_a(t)$ be the entry flow and exit flow, respectively. The arc entry/exit flows are the sum of entry/exit flows associated with individual paths using this arc; that is

$$u_a(t) \;=\; \sum_{p\in\mathcal{P}}\delta_{ap}\,u_a^p(t), \qquad w_a(t) \;=\; \sum_{p\in\mathcal{P}}\delta_{ap}\,w_a^p(t), \qquad \forall\; a\in A \tag{4.17}$$

where

$$\delta_{ap} \;=\; \begin{cases} 1 & \text{if arc } a \text{ belongs to path } p \\ 0 & \text{otherwise} \end{cases}$$

Let us also define the cumulative entering vehicle count $U_a(t)$ and cumulative exiting vehicle count $W_a(t)$, for each arc a. Similarly, each one is disaggregated into quantities associated with each path that uses this arc:

$$U_a(t) \;=\; \sum_{p\in\mathcal{P}}\delta_{ap}\,U_a^p(t), \qquad W_a(t) \;=\; \sum_{p\in\mathcal{P}}\delta_{ap}\,W_a^p(t), \qquad \forall\; a\in A \tag{4.18}$$

The arc delay $D_a(t)$ is the time taken to traverse the arc a when the time of entry is t. The arc exit time function $\tau_a(t)$ is defined as $\tau_a(t)\doteq t+D_a(t)$, that is, $\tau_a(t)$ represents the exit time on arc a when the time of entry is t.

For each group of drivers using the arc, the ratio of their arrival and departure rates must be the same under FIFO. This is expressed as

$$w_a^p\big(\tau_a(t)\big) \;=\; \begin{cases} w_a\big(\tau_a(t)\big)\cdot\dfrac{u_a^p(t)}{u_a(t)}, & \text{if } \; u_a(t)\;\neq\;0 \\ 0 & \text{if } \; u_a(t)\;=\;0 \end{cases} \tag{4.19}$$

(4.19) uniquely determines the turning percentages at junctions with more than one outgoing link, and is consistent with the FIFO discipline and established route choices. It remains to express the path delay as the sum of finitely many link delays. If we describe path $p\in\mathcal{P}$ as the following sequence of conveniently labeled arcs:

$$p \;=\; \big\{a_1,\,a_2,\,\ldots,\,a_{i-1},\,a_i,\,a_{i+1}\,\ldots,\,a_{m(p)}\big\}$$

where $m(p)$ is number of arcs in path p.

It then follows immediately that path arrival time along p when the departure time at origin is t can be expressed as a composition of arc exit time functions:

$$\tau^p(t) \;=\; \tau_{a_{m(p)}}\circ\ldots\circ\tau_{a_2}\circ\tau_{a_1}(t) \qquad p \;=\; \big\{a_1,\,a_2,\,\ldots,\,a_{m(p)}\big\}\in\mathcal{P} \tag{4.20}$$

where the operator $\circ$ means decomposition, that is, $f \circ g(x) \equiv f\big(g(x)\big)$.

Now the complete network loading procedure is given by (4.14)-(4.20), which is interpreted as a well defined *differential algebraic equation* (DAE) system. Moreover, as well shall see in the next section, the (effective) path delay operator defined in this way is strongly continuous from the subset Λ of a Hilbert space into $\Big(\mathcal{L}^2[t_0,\, t_f]\Big)^{|\mathcal{P}|}$.

5 Existence of the DUE

Existence results for DUE are most general if based on formulation (3.10). Theorem 2.5 for the existence of solutions of variational inequalities in topological spaces can be applied if the operators $\Psi_p(t,\,\cdot)$ can be shown to be continuous and the feasible set Λ can be shown to be compact. After Section 5.1 addresses the continuity of the effective delay operator, based on the DNL model introduced in previously, the last obstacle to proving existence is the compactness of Λ, which unfortunately does not generally occur in SRDC DUE. To overcome the aforementioned difficulty, we will consider instead successive finite-dimensional approximations of Λ, and rely on a topological argument. Such an approach is mathematically rigorous but much more challenging than would be the case if Λ were compact in the appropriate Hilbert space. The topological argument and supporting infrastructure for a proof of existence are presented in Section 5.2 and Section 5.3.

5.1 Continuity of the effective path delay operator

In this section, we will establish continuity of the map $h \mapsto \Psi(\cdot,\, h)$. These results will be essential for the proof of existence theorem for DUE in Section 5.3. Notice that unlike the argument in Zhu and Marcotte (2000) which required *a priori* bound for the path flows, the proof provided here works for unbounded path flows and even distributions, thanks to the generalized Vickrey model.

The next lemma provides the criterion for the continuity of the delay function $D_a(\cdot)$, $a \in A$.

Lemma 5.1. *Consider an arc $a \in A$, with inflow $u_a(\cdot)$. Under the generalized Vickrey model expressed in (4.14)-(4.16), the arc delay function $D_a(\cdot)$ is continuous if $u_a(\cdot) \in \mathcal{L}^2[t_0,\, t_f]$.*

Proof. Assume that $u_a(\cdot) \in \mathcal{L}^2[t_0,\, t_f]$, then $u_a(\cdot) \in \mathcal{L}^1[t_0,\, t_f]$. Therefore the cumulative entering vehicle count

$$U_a(t) \doteq \int_{t_0}^{t} u_a(s)\, ds$$

is absolutely continuous. It is straightforward to verify that the following quantity is continuous.

$$q_a(t) \doteq U_a(t) - M_a\, t - \min_{\tau \leq t - T_a} \{U_a(\tau) - M_a\, \tau\}$$

where $q_a(t)$ denotes the queue length, M_a, T_a denotes flow capacity and free flow time, respectively. By (4.16), the delay function $D_a(\cdot)$ is continuous. $\square$

The next lemma is a technical result that will facilitate the proof of Theorem 5.3.

Lemma 5.2. *Assume $f(\cdot) : [a_2,\, b_2] \to \Re^1$ is continuous. Let $g_n(\cdot) : [a_1,\, b_1] \to [a_2,\, b_2]$, $n \geq 1$ be a sequence of functions such that g_n converges to $g(\cdot) : [a_1,\, b_1] \to [a_2,\, b_2]$ uniformly. Then*

$$f\big(g_n(\cdot)\big) \longrightarrow f\big(g(\cdot)\big), \qquad n \longrightarrow \infty$$

uniformly.

Proof. According to the Heine-Cantor theorem, $f(\cdot)$ is uniformly continuous on $[a_2, b_2]$. It follows that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2 \in [a_2, b_2]$, whenever $|y_1 - y_2| < \delta$, the inequality

$$|f(y_1) - f(y_2)| \leq \varepsilon$$

holds. Moreover, by the uniform convergence of g_n, there exists some $N > 0$ such that, for all $n > N$, we have

$$|g_n(x) - g(x)| < \delta, \qquad \forall\, x \in [a_1, b_1]$$

Thus, for every $n > N$,

$$\big|f\big(g_n(x)\big) - f\big(g(x)\big)\big| \leq \varepsilon, \qquad \forall\, x \in [a_1, b_1]$$

□

Theorem 5.3. *Under the network loading model described in Section 4, the effective path delay operator* $\Psi(t, \cdot) : \Lambda \to \big(\mathcal{L}^2([t_0, t_f])\big)^{|\mathcal{P}|}$, $h \mapsto \Psi(\cdot, h)$ *is well-defined and continuous.*

Proof. For each $h \in \Lambda$, the functions $\Psi_p(\cdot, h)$ for all $p \in \mathcal{P}$ are uniquely determined by the network loading procedure. To show that the effective path delay operator is well-defined, it remains to show that $\Psi(\cdot, h) \in \Big(\mathcal{L}^2[t_0, t_f]\Big)^{|\mathcal{P}|}$ for each $h \in \Lambda$. Notice that there exists an upper bound for the path delays regardless of the network flow profile:

$$D_p(t, h) \leq \sum_{a \in p} \left\{ \frac{1}{M_a} \sum_{(i,j) \in \mathcal{W}} Q_{ij} + T_a \right\}, \qquad \forall\, h \in \Lambda,\, p \in \mathcal{P},\, t \in [t_0, t_f]$$

where M_a, T_a are the flow capacity and free flow travel time respectively, that are associated with the arc a. Recall the definition of effective path delay (3.6):

$$\Psi_p(t, h) = D_p(t, h) + \mathcal{F}\Big(t + D_p(t, h) - T_A\Big)$$

Since $\mathcal{F}(\cdot)$ is continuous, the uniform boundedness of $D_p(t, h)$ thus implies the uniform boundedness of $\Psi_p(t, h)$ for all $h \in \Lambda$, $p \in \mathcal{P}$ and $t \in [t_0, t_f]$. This leads to the conclusion that $\Psi(\cdot, h) \in \Big(\mathcal{L}^2[t_0, t_f]\Big)^{|\mathcal{P}|}$ for all $h \in \Lambda$. With the preceding as background, the proof of continuity of the effective delay operator may be given in five parts.

Part 1. Consider first only a single link with a sequence of entry flows u_ν where $\nu \geq 1$ that converge to u in the $\mathcal{L}^2$ norm. That is.

$$\|u_\nu - u\|_2 \doteq \left(\int_{t_0}^{t_f} \big(u_\nu(t) - u(t)\big)^2\, dt\right)^{1/2} \longrightarrow 0 \quad \text{as} \quad \nu \longrightarrow \infty$$

Consider the cumulative entering vehicle counts

$$\begin{aligned} U_\nu(t) &\doteq \int_{t_0}^{t} u_\nu(s)\, ds \qquad \nu \geq 1 \\ U(t) &\doteq \int_{t_0}^{t} u(s)\, ds \end{aligned} \qquad\qquad t \in [t_0, t_f]$$

then U_ν converge to U uniformly on $[t_0, t_f]$. This is due to the simple observation

$$|U_\nu(t) - U(t)| \leq \int_{t_0}^{t} |u_\nu(s) - u(s)|\, ds \leq \|u_\nu - u\|_1 \leq (t_0 - t_f)^{1/2} \|u_\nu - u\|_2 \longrightarrow 0$$

where $\|\cdot\|_1$ is the norm in $\mathcal{L}^1[t_0,\, t_f]$. The last inequality is a version of Jenssen's inequality.
Part 2. Define $R(\tau) \doteq U(\tau) - M\,\tau$, $R_\nu(\tau) \doteq U_\nu(\tau) - M\,\tau$. We claim the following uniform convergence holds

$$\min_{\tau\leq t}\{R_\nu(\tau)\} \;\longrightarrow\; \min_{\tau\leq t}\{R(\tau)\} \qquad \forall\, t\in[t_0,\,t_f] \tag{5.21}$$

where M is the link flow capacity. Indeed, for any $\varepsilon>0$, by uniform convergence of U_ν, we can choose N such that for all $\nu\geq N$, the following inequality holds

$$|U_\nu(t)-U(t)| \;\leq\; \varepsilon \qquad \forall\, t\in[t_0,\,t_f]$$

For any fixed t, if $\nu\geq N$, we have

$$|R_\nu(\tau)-R(\tau)| \;=\; |U_\nu(\tau)-U(\tau)| \;\leq\; \varepsilon \tag{5.22}$$

Let $\hat{\tau} = \text{argmin}_{\tau\leq t}\{R(\tau)\}$, so that, by (5.22), we have

$$\min_{\tau\leq t}\{R_\nu(\tau)\} \;\leq\; R_\nu(\hat{\tau}) \;\leq\; R(\hat{\tau})+\varepsilon \;=\; \min_{\tau\leq t}\{R(\tau)\}+\varepsilon \tag{5.23}$$

On the other hand, let $\hat{\tau}_\nu = \text{argmin}_{\tau\leq t}\{R_\nu(\tau)\}$. Then for each $\nu\geq N$, it must be that

$$\min_{\tau\leq t}\{R(\tau)\} \;\leq\; R(\hat{\tau}_\nu) \;\leq\; R^{(\nu)}(\hat{\tau}_\nu)+\varepsilon \;=\; \min_{\tau\leq t}\{R_\nu(\tau)\}+\varepsilon \tag{5.24}$$

Taken together, (5.23) and (5.24) imply

$$\left|\min_{\tau\leq t}\{R_\nu\} - \min_{\tau\leq t}\{R(\tau)\}\right| \;\leq\; \varepsilon, \qquad \forall\,\nu\geq N$$

Since t is arbitrary, the claim is demonstrated.
Part 3. The immediate consequence of **Part 2** and (4.14)-(4.16) is the following uniform convergence

$$W_\nu(t) \longrightarrow W(t), \quad q_\nu(t) \longrightarrow q(t), \quad D_\nu(t) \longrightarrow D(t), \quad \tau_\nu(t) \longrightarrow \tau(t), \quad \nu \longrightarrow \infty \tag{5.25}$$

for which we employ notation whose meaning is transparent. The next step is to extend such convergence to the whole network. Consider the sequence of departure rates h_ν converging to h in the $\|\cdot\|_{\mathcal{L}^2}$ norm. By the definition (3.5), this implies each path flow $h_{p,\nu}(\cdot)\to h_p$ in the $\|\cdot\|_2$ norm, for all $p\in\mathcal{P}$. A simple induction based on results established in **Part 2** yields, as $\nu\longrightarrow\infty$,

$$U_{a,\nu}(t) \longrightarrow U_a(t), \quad W_{a,\nu}(t) \longrightarrow W_a(t), \quad D_{a,\nu}(t) \longrightarrow D_a(t), \quad \tau_{a,\nu}(t) \longrightarrow \tau_a(t), \tag{5.26}$$

uniformly for all $a\in A$.
Part 4. We will show next the uniform convergence of the path delay function $D_p(\cdot,\,h_\nu)\to D_p(\cdot,\,h)$, based on (5.26). Recall the path exit time function (4.20)

$$\tau^p(t) \;=\; \tau_{a_{m(p)}}\circ\ldots\circ\tau_{a_2}\circ\tau_{a_1}(t) \qquad p \;=\; \left\{a_1,\,a_2,\,\ldots,\,a_{m(p)}\right\}\in\mathcal{P} \tag{5.27}$$

The desired uniform convergence of path delay follows from the uniform convergence $\tau^p_\nu(t)\to\tau^p(t)$. We start by showing that $\tau_{a_2,\nu}\circ\tau_{a_1,\nu}(t)\to\tau_{a_2}\circ\tau_{a_1}(t)$ uniformly

For every $\nu\geq 1$, since the inflow of arc a_2 is square-integrable, $\tau_{a_2,\nu}(\cdot)$ is continuous by Lemma 5.1. This means that $\tau_{a_2}(\cdot)$ is also continuous since it is the uniform limit of $\tau_{a_2,\nu}(\cdot)$.

Lemma 5.2 then implies that $\tau_{a_2}\big(\tau_{a_1,\nu}(\cdot)\big)$ converges uniformly to $\tau_{a_2}\big(\tau_{a_1}(\cdot)\big)$, that is, for any $\varepsilon > 0$, there exists an $N_1 > 0$ such that for all $\nu > N_1$,

$$\left|\tau_{a_2}\big(\tau_{a_1,\nu}(t)\big) - \tau_{a_2}\big(\tau_{a_1}(t)\big)\right| \;<\; \varepsilon/2, \qquad \forall\, t \in [t_0,\, t_f]$$

Moreover, there exists some $N_2 > 0$ such that for all $\nu > N_2$,

$$\left|\tau_{a_2,\nu}(t) - \tau_{a_2}(t)\right| \;<\; \varepsilon/2, \qquad \forall\, t \in [t_0,\, t_f]$$

Now let $N_0 = \max\{N_1,\, N_2\}$. For any $\nu > n$, and all $t \in [t_0,\, t_f]$,

$$\begin{aligned}
& \left|\tau_{a_2,\nu}\big(\tau_{a_1,\nu}(t)\big) - \tau_{a_2}\big(\tau_{a_1}(t)\big)\right| \\
\leq\; & \left|\tau_{a_2,\nu}\big(\tau_{a_1,\nu}(t)\big) - \tau_{a_2}\big(\tau_{a_1,\nu}(t)\big)\right| + \left|\tau_{a_2}\big(\tau_{a_1,\nu}(t)\big) - \tau_{a_2}\big(\tau_{a_1}(t)\big)\right| \\
<\; & \varepsilon/2 + \varepsilon/2 \;=\; \varepsilon
\end{aligned}$$

This shows the desired uniform convergence $\tau_{a_2,\nu} \circ \tau_{a_1,\nu}(t) \to \tau_{a_2} \circ \tau_{a_1}(t)$.

The uniform convergence $\tau_\nu^p(\cdot) \to \tau^p(\cdot)$ follows immediately by (5.27) and induction. As a result, we obtain the uniform convergence of path delay

$$D_p(\cdot,\, h_\nu) \;\longrightarrow\; D_p(\cdot,\, h), \qquad \nu \;\longrightarrow\; \infty$$

Part 5. Finally, recall the definition of the effective delay

$$\Psi(t,\, h) \;=\; D_p(t,\, h) + \mathcal{F}\big(t + D_p(t,\, h) - T_A\big)$$

Note that $\mathcal{F}(\cdot)$ is continuous, the following uniform convergence follows by Lemma 5.2

$$\mathcal{F}\big(t + D_p(t,\, h_\nu) - T_A\big) \;\longrightarrow\; \mathcal{F}\big(t + D_p(t,\, h) - T_A\big), \qquad \nu \;\longrightarrow\; \infty$$

We conclude that the effective delay $\Psi_p(\cdot,\, h_\nu)$ converges uniformly to $\Psi_p(\cdot,\, h)$. The desired convergence in the $\|\cdot\|_{\mathcal{L}^2}$ norm now follows since the interval $[t_0,\, t_f]$ is compact. $\square$

5.2 Alternative definition of effective path delay

The integrals employed in defining the feasible domain (3.8) are not enough to assure bounded path flows h_p, $p \in \mathcal{P}$. This observation is the fundamental hurdle to providing existence of the DUE solution. One of the main accomplishments of this paper is to address the boundedness of path flows not only for the proof of existence result but also for future analysis and estimation of network flows. In this section, we will present an alternative formulation of the effective path delay $\Psi_p(t,\, h)$, where that alternative formulation will facilitate our analysis leading to the proof of our main result, Theorem 5.6.

Recall the effective delay operator

$$\Psi_p(t,\, h) \;\doteq\; D_p(t,\, h) + \mathcal{F}\big(t + D_p(t,\, h) - T_A\big) \tag{5.28}$$

In order to simplify our analysis, it is convenient to rewrite (5.28) in a slightly different form. In particular, for each O-D pair $(i,\, j) \in \mathcal{W}$, let us introduce the cost functions $\phi_{ij}(\cdot) : [t_0,\, t_f] \to \Re^1_+$ and $\psi_{ij}(\cdot) : [t_0,\, t_f] \to \Re^1_+$, which measure the travel costs at the origin and destination of the path, respectively. More precisely, $\phi_{ij}(\cdot)$ is a function of departure time, while $\psi_{ij}(\cdot)$ is a

function of arrival time. Fix any vector of path flows $h \in \Lambda$, recall the path exit time function $\tau^p(t) = t + D_p(t,\, h)$. Then (5.28) can be equivalently written as

$$\begin{aligned}\Psi_p(t,\, h) &= -t + \tau^p(t) + \mathcal{F}\big(\tau^p(t) - T_A\big) \\ &= \phi_{ij}(t) + \psi_{ij}\big(\tau_p(t,\, h)\big)\end{aligned}$$

where

$$\phi_{ij}(t) \equiv -t, \qquad \psi_{ij}(t) \doteq t + \mathcal{F}\big(t - T_A\big) \tag{5.29}$$

Remark 5.4. *In Bressan and Han (2011, 2012), the travel costs are measured in terms of $\phi_{ij}(\cdot)$, $\psi_{ij}(\cdot)$, in other words, the general effective delay (5.28) can be alternatively evaluated as a sum of costs at the beginning and end of journey of each driver.*

In Section 5.3, we will exploit the alternative representative of effective path delay (cost) to establish existence.

To prepare for the existence proof, consider a general network $G(N,\, A)$. Associate with each O-D pair $(i,\, j) \in \mathcal{W}$ the pair of cost functions $\phi_{ij}(\cdot)$ and $\psi_{ij}(\cdot)$ as defined in (5.29). We make the following two assumptions on $\phi_{ij}(\cdot)$, $\psi_{ij}(\cdot)$ and the underlying traffic flow model.

A1. *For all, $\phi_{ij}(\cdot)$, $\psi_{ij}(\cdot)$ is continuously differentiable on $[t_0,\, t_f]$. In addition,*

$$\frac{d}{dt}\psi_{ij}(t) > 0, \qquad t \in [t_0,\, t_f] \tag{5.30}$$

A2. *Each link $a \in A$ in the network has a flow capacity $f_a^{max} < \infty$*

Note that A1 stipulates that each arrival cost $\psi_{ij}(\cdot)$ is strictly increasing, while A2 applies to all traffic flow models that assume a flow capacity for each network link. Notice that by (5.29), A1 amounts to requiring that

A1′. *$\mathcal{F}(\cdot)$ is continuously differentiable and satisfies $\mathcal{F}'(s) > -1$ on $[t_0,\, t_f]$.*

Remark 5.5. *In view of the preceding assumptions, we are prompted to define the following:*

$$\psi'_{min} \doteq \min_{(i,\,j)\in\mathcal{W}}\ \min_{t\in[t_0,\,t_f]} \frac{d}{dt}\psi_{ij}(t) > 0 \tag{5.31}$$

$$F^{max} \doteq \max_{a\in A} f_a^{max} < +\infty \tag{5.32}$$

Note that $\psi'_{min} > 0$ follows from (5.30) and the fact that $\frac{d}{dt}\psi_{ij}(\cdot)$ is continuous on the compact interval $[t_0,\, t_f]$.

5.3 Existence of solution to the variational inequality

The classical result explained by Theorem 2.5 will be the key ingredient for the proof of existence of the DUE solution. Using the same notation as in Theorem 2.5, the underlying topological vector space E will be instantiated by $\big(\mathcal{L}^2([t_0,\, t_f])\big)^{|\mathcal{P}|}$, which is a locally convex topological vector space. The dual space E^* will be again $\big(\mathcal{L}^2([t_0,\, t_f])\big)^{|\mathcal{P}|}$.

Theorem 5.6. ***(Existence of DUE)*** *Let assumption (A1'), (A2) hold. In addition, assume that the effective delay operator* $\Psi : \Lambda \to \big(\mathcal{L}^2[t_0,\, t_f]\big)^{|\mathcal{P}|}$ *is continuous. Then the dynamic user equilibrium problem as in Definition 3.1 has a solution.*

Proof. The proof is divided into four parts.

Part 1. Our strategy for demonstrating existence is to adapt Theorem 2.5 to the locally convex topological vector space $\big(\mathcal{L}^2([t_0,\, t_f])\big)^{|\mathcal{P}|}$, and its subset Λ. By assumption, the map $h \mapsto \Psi(\cdot,\, h)$ is continuous from Λ to the space of $\big(\mathcal{L}^2([t_0,\, t_f])\big)^{|\mathcal{P}|}$. If Λ were compact and convex, we would immediately demonstrate the desired result. However, Λ is bounded, closed and convex, but not compact in $\big(\mathcal{L}^2([t_0,\, t_f])\big)^{|\mathcal{P}|}$.

Part 2. We will instead employ finite-dimensional approximations of Λ. In order to proceed, consider for each $n \geq 1$ the uniform partition of interval $[t_0,\, t_f]$ with 2^n sub-intervals

$$\begin{aligned} t_0 \;&=\; t^0 \;<\; t^1 \;<\; t^2 \;\ldots\; <\; t^{2^n} \;=\; t_f \\ t^i - t^{i-1} \;&=\; \frac{t_f - t_0}{2^n} \qquad i = 1, \ldots, 2^n \end{aligned}$$

Then consider the following sequence of finite-dimensional subsets

$$\Lambda_n \;\doteq\; \big\{h \in \Lambda : \quad h_p(\cdot) \text{ is constant on } [t^{i-1},\, t^i), \quad p \in \mathcal{P}\big\} \;\subset\; \Lambda \tag{5.33}$$

We claim that for each $n \geq 1$, Λ_n is compact and convex in $\big(\mathcal{L}^2([t_0,\, t_f])\big)^{|\mathcal{P}|}$. Indeed, given any $h^{n,1},\, h^{n,2} \in \Lambda_n$, and $\alpha \in [0,\, 1]$, $\alpha\, h^{n,1} + (1-\alpha)\, h^{n,2}$ is clearly nonnegative and constant on each $[t^{i-1},\, t^i]$ for $i = 1, \ldots, 2^n$. In addition, for any origin-destination pair $(i,\, j) \in \mathcal{W}$, by definition (3.8),

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \alpha\, h^{n,1}(t) + (1-\alpha)\, h^{n,2}(t)\, dt \;=\; \alpha\, Q_{ij} + (1-\alpha)\, Q_{ij} \;=\; Q_{ij}$$

This verifies that Λ_n is convex. To see compactness, we define the map $\mu : \Lambda_n \to \Re_+^{2^n \times |\mathcal{P}|}$, $h \mapsto (a_{i,\,p} :\ 1 \leq i \leq 2^n,\ p \in \mathcal{P})$ where each vector $(a_{1,p}, \ldots, a_{2^n,p})$ is the coordinate of h_p under the natural basis $\{e_n^i\}_{i=1}^{2^n}$, where

$$e_n^i(t) \;=\; \begin{cases} 1 & \quad t \in [t^{i-1},\, t^i) \\ 0 & \quad \text{else} \end{cases}$$

Clear, the map μ is one-to-one. Now consider any sequence $\{h^{n(\nu)}\}_{\nu \geq 1} \subset \Lambda_n$, and the sequence of their images $\{\mu\big(h^{n(v)}\big)\}_{\nu \geq 1} \subset \Re_+^{2^n \times |\mathcal{P}|}$. By (3.8), $\{\mu(h^{n(\nu)}\}_{\nu \geq 1}$ is uniformly bounded by the following quantity

$$\max_{i,j} \frac{2^n\, Q_{ij}}{t_f - t_0}$$

By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\big\{\mu(h^{n(\nu')})\big\}_{\nu' \geq 1}$. By construction, the subsequence $\big\{h^{n(\nu')}\big\}_{\nu' \geq 1}$ must converge uniformly to some $\hat{h}^n$. In view of the compact interval $[t_0,\, t_f]$, we conclude that this convergence is also in the $\|\cdot\|_{\mathcal{L}^2}$ norm. It now remains to show $\hat{h}^n \in \Lambda_n$, then our claim follows from sequential compactness of Λ_n. Clearly $\hat{h}^n \geq 0$ and is constant on the sub-intervals $[t^{i-1},\, t^i)$, $i = 1, \ldots, 2^n$. Moreover, since $h^{n(\nu')}$ are uniformly bounded, by the Dominated Convergence Theorem,

$$Q_{ij} \;=\; \lim_{\nu' \to \infty} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^{n(\nu')}(t)\, dt \;=\; \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \hat{h}_p^n(t)\, dt$$

This implies $\hat{h}^n \in \Lambda_n$. Thereby, the claim is substantiated.

Part 3. For each $n \geq 1$, apply Theorem 2.5 to Λ_n and obtain $h^{n,*} \in \Lambda_n$ such that

$$\Big\langle \Psi\big(\cdot,\, h^{n,*}\big),\, h^{n,*}(\cdot) - h^n(\cdot) \Big\rangle \;\geq\; 0, \qquad \forall\, h^n \in \Lambda_n \tag{5.34}$$

Where the $\langle\, ,\, \rangle$ is the inner product (duality) defined in (3.4). It is easy to observe that (5.34) implies that if $h_p^{n,*}(t) > 0$, $t \in [t^j,\, t^{j+1}]$, then

$$\int_{t^j}^{t^{j+1}} \Psi(t,\, h^{n,*})\, dt \;=\; \min_{0 \leq k \leq 2^n} \int_{t^k}^{t^{k+1}} \Psi(t,\, h^{n,*})\, dt \tag{5.35}$$

Recall (5.31), (5.32), choose any constant $\mathcal{M}$ such that

$$\mathcal{M} \;>\; \frac{3F^{max}}{\psi'_{min}} \tag{5.36}$$

We then claim that for all $n \geq 1$, there must hold $h_p^{n,*}(t) \leq \mathcal{M}$ for all $t \in [t_0,\, t_f]$, $p \in \mathcal{P}$. Otherwise, assume there exists some $p \in \mathcal{P}$, some $\nu \geq 1$ and some $0 \leq i \leq 2^\nu$ with

$$h^{v,*}(t) \;\equiv\; \mu \;>\; \mathcal{M} \qquad t \in [t^i,\, t^{i+1}],$$

By choosing $[t_0,\, t_f]$ large enough, we can assume that $i \geq 1$. Now consider the interval $[t^{i-1},\, t^i]$, and the quantity $\sup\limits_{t \in [t^{i-1}, t^i]} \Psi(t,\, h^{v,*})$, by possibly modifying the value of the function $\Psi(\cdot,\, h^{v,*})$ at one point, we can obtain $t^* \in [t^{i-1},\, t^i]$ such that

$$\Psi(t^*,\, h^{v,*}) \;=\; \sup_{t \in [t^{i-1}, t^i]} \Psi(t,\, h^{v,*})$$

Now, let $\tau_p(t,\, h^{v,*}) = t + D_p(t,\, h^{v,*})$ be the arrival time function. According to FIFO and assumption *(A2)*, we deduce that for all $t \in [t^i,\, t^{i+1}]$,

$$\tau_p(t,\, h^{v,*}) - \tau_p(t^*,\, h^{v,*}) \;\geq\; \frac{(t - t^i)\,\mu}{F^{max}} \tag{5.37}$$

this implies, together with (5.31) that

$$\psi_{ij}\big(\tau_p(t,\, h^{v,*})\big) - \psi_{ij}\big(\tau_p(t,\, h^{v,*})\big) \;\geq\; \psi'_{min}\big(\tau_p(t,\, h^{v,*}) - \tau_p(t^*,\, h^{v,*})\big) \;\geq\; \psi'_{min} \cdot \frac{(t - t^i)\,\mu}{F^{max}} \tag{5.38}$$

(5.38) and (5.29) imply

$$\Psi_p(t,\, h^{v,*}) - \Psi_p(t^*,\, h^{v,*}) \;\geq\; \frac{\psi'_{min}\,\mu}{F^{max}}(t - t^i) - (t - t^*), \qquad \forall\, t \in [t^i,\, t^{i+1}] \tag{5.39}$$

Integrating (5.39) from t^i to t^{i+1} and a simple calculation yield

$$\int_{t^i}^{t^{i+1}} \Psi(t,\, h^{v,*})\, dt - (t^{i+1} - t^i)\Psi(t^*,\, h^{v,*}) \;\geq\; \frac{(t^{i+1} - t^i)^2}{2} \cdot \frac{\psi'_{min}\,\mu}{F^{max}} + \frac{(t^{i+1} - t^i)}{2} \cdot \left(t^* - \frac{t^i + t^{i+1}}{2}\right) \tag{5.40}$$

Notice since $t^* \in [t^{i-1},\, t^i]$, we deduce $t^* - (t^i + t^{i+1})/2 \geq -3/2(t^{i+1} - t^i)$. (5.40) then becomes

$$\int_{t^i}^{t^{i+1}} \Psi(t,\, h^{v,*})\, dt - (t^{i+1} - t^i)\Psi(t^*,\, h^{v,*}) \;\geq\; \frac{(t^{i+1} - t^i)^2}{2} \left(\frac{\psi'_{min}\,\mu}{F^{\max}} - 3\right) \;>\; 0 \tag{5.41}$$

This yields the following contradiction to (5.35) and hence (5.34)

$$\int_{t^i}^{t^{i+1}} \Psi(t,\, h^{v,*})\, dt \;>\; \int_{t^{i-1}}^{t^i} \Psi_p(t,\, h^{v,*})\, dt$$

Part 4. By previous steps, we have obtained a uniformly bounded sequence of vector-valued functions $\left\{h_n^*\right\}_{n\geq 1}$ satisfying (5.34). By taking a subsequence, we can assume the weak convergence in both $\mathcal{L}^2$-space and $\mathcal{L}^1$-space

$$h^{n,*} \;\longrightarrow\; h^*, \qquad n \;\to\; \infty$$

for some $h^* \in \Lambda$. We claim that for all $h \in \Lambda$,

$$\Big\langle \Psi\big(\cdot,\, h^*\big),\, h^*(\cdot) - h(\cdot) \Big\rangle \;\geq\; 0$$

Indeed, given any $h \in \Lambda$, there exists piecewise-constant approximation $\{h_n \in \Lambda_n\}_{n\geq 1}$, which converges to h both point-wise and in the $\|\cdot\|_{\mathcal{L}^2}$ norm. According to (5.34), we have

$$\Big\langle \Psi_p\big(\cdot,\, h^{n,*}\big),\, h^{n,*}(\cdot) - h_n(\cdot) \Big\rangle \;\geq\; 0 \tag{5.42}$$

Notice that the map $h \mapsto \Psi_p(\cdot,\, h)$ is continuous with respect to the $\|\cdot\|_{\mathcal{L}^2}$ norm, by the continuity of the inner product, we pass (5.42) to the limit

$$\Big\langle \Psi_p\big(\cdot,\, h^*\big),\, h^*(\cdot) - h(\cdot) \Big\rangle \;\geq\; 0$$

□

The next theorem establishes the existence of SRDC DUE with generalized Vickrey model, which is an immediate consequence of Theorem 5.3 and Theorem 5.6.

Theorem 5.7. *Let assumption (A1') hold, then the dynamic user equilibrium as in Definition 3.1 with generalized Vickrey model has a solution.*

Remark 5.8. *Notice that the assumption A2 is satisfied by the Vickrey's model and hence the GVM. Thus in the statement of the theorem, A2 is omitted.*

6 Conclusion

We have established the existence of the continuous-time simultaneous route-and-departure choice DUE for the generalized Vickrey model (GVM), a generalization of Vickrey's model, and plausible regularity conditions that are easy to check and rather weak. It is significant that ours is the first DUE existence result without the a priori bounding of departure rates (path flows) and the most general constraint relating path flows to the trip table. In fact, our method of proof successfully overcomes two major hurdles that have stymied other researchers:

1. the set of feasible flows Λ is intrinsically non-compact in $\mathcal{L}^2$-space as well as $\mathcal{L}^1$-space; and

2. a direct topological argument requires *a priori* bounds for the path flows, where those bounds do not arise from any behavioral argument or theory.

Theorem 5.6 is a general result that subsumes all SRDC DUE models regardless of the arc dynamics, flow propagation and arc delay function employed, as long as the effective delay operator is continuous.

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